Finite Relativist Geometry Grounded in Perceptual Operations

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Abstract. Formal geometry is a fundamental tool for showing how relevant metric qualities, such as depths, lengths, and volumes, as well as location concepts, such as points, can be constructed from experience. The ontological challenge of information grounding lies in the choice of concepts to consider as primitive, vs. those to be constructed. It also lies in accounting for the relativity and finiteness of experiential space. The grounding approach proposed here constructs geometrical concepts from primitives of the *human attentional apparatus* for guiding attention and performing *perceptual operations*. This apparatus enables humans to take attentional steps in their perceived vista environment and to perform geometric comparisons. We account for the relativity of experienced space by constructing locations relative to a reference frame of perceived *point-like features*. The paper discusses perceptual operations and the idea of point-like features, and introduces a constructive calculus that reflects the generation of domains of geometric comparison from the perspective of an observer. The calculus is then used to construct a model and to motivate an axiomatization of absolute geometry in a finite relativist flavour.

Keywords: constructive Euclidean geometry, relativist geometry, information grounding, operational semantics

1 Introduction

How should the multitude of spatial concepts underlying spatial data be interpreted in terms of experience? The philosophical grounding problem [12] gains practical relevance if we ask ourselves what kind of observations a certain data set refers to [36,16]. This question has recently led scientists to regard sensors in a wider sense, including human observers, as a means to ground the semantic web [14]. From a practical viewpoint, it remains often unclear how geometrical attributes like widths, heights, depths and directions were (or could have been) practically obtained. A waterbody gives rise to many possible water depths if the underlying reference operations remain hidden [30,37]. Likewise, a given location can be conceived in many different ways: relative to diverse egocentric or allocentric reference frames [25], as well as in terms of geographic coordinates. Since locations and geometric attributes are among the major categories underlying spatial data, we are interested in the kinds of inter-subjective space experiences they originate from, similar to [23].

Geometry, as all traditional mathematics, evolved from concrete experiences and problems, stated by the Greeks or earlier. In order to achieve greater generality, modern age mathematicians drove their discipline away from these experiences by means of *abstraction* and *domain closure*. For example, in arithmetics, the experiential basis of natural numbers in counting soon became extended in order to incorporate infinity, rational, real, and complex numbers. Just as arithmetics abandoned explicit counting operations in order to close the numbers with respect to arithmetic operations, geometry closed its experiential domain of measurement by assuming infinities of points.

From the perspective of information grounding, the undisputed merits of domain closure and abstraction make it sometimes difficult to see what the roots of geometric information are. These roots of information play an important role in all kinds of semantic reference systems [16]. In particular, spatial reference systems are established by geodesists in terms of observed directions, angles and lengths related to physically real, not abstract, phenomena. Thus, the kind of geometry performed by a geodesist is different from mathematical geometry in that it is *constructive* and *finite* instead of abstract and infinite[1]. This remains true even if calculations are performed on discrete approximations of real number fields, as they are in computers. Axiomatic geometries commonly begin with abstractions but do not account for how they are obtained. They populate their universe of discourse with abstract points, spheres, lines and planes, even though an infinitely small point is a mental fiction [48].

Furthermore, from a practical viewpoint, there is no way of determining an absolute point in space and time[2]. This *relativity of space* was recognized already by Leibniz [18], but manifests itself also in spatial cognition research. Different egocentric and allocentric frames of reference serve to construct different levels of space apprehension: Starting from the space around the body [47], we arrive at navigation space by reconstruction from memory [25]. It seems therefore inappropriate to base a theory about grounding spatial information on the assumption of absolute or abstract space.

In this paper, we propose to conceive of geometry in terms of perceptual operations, namely perceptual predications on foci of attention, as first suggested in [37]. Foci of attention are atomic (but finite) moments in which an observer's attention is focused on some spot in his vista environment. Our predications are

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1 For similar reasons, Habel [11] has proposed that cognitively adequate temporal reference systems should be *finite* with a so-called *density in intensio*. A similar idea stands behind Aristotle's notion of *potential infinities* [1]. In our view, potentiality can only mean that repeatable operations are available, following [20].

2 Even if we use a *spatial reference system*, this system is logically anchored in (and therefore presupposes the identity of) concrete places. Such an anchor place is a necessary part of a *geodetic datum* for a mathematical ellipsoid representing the earth surface.

inter-subjectively available operations for comparing foci[3]. The current paper adds to this idea by accounting for relativism and constructive finitism of experiential geometry (Sect. 2). We provide perceptual justification for our choice of primitives and argue that the identity of locations needs to be constructed based on length and direction comparisons taken with reference to some anchor frame consisting of point-like features, such as a particular end of a rod (Sect. 3). We then introduce an operational calculus that allows us to generate an appropriate finite model. Its rules can be used to motivate axioms of finite relativist geometry.

2 Constructive and Axiomatic Geometries

What kinds of mathematical entities should be presumed in order to account for experiential geometry? Axiomatic geometry is not a one-way street. Since Euclid's elements, a number of formalizations of Euclidean geometry have been proposed, with different assumptions about admitted objects and relations. For example, Hilbert [13] presupposed lines and points, whereas Tarski presumed only points and two kinds of relations on them [44]. Pointless geometries [43,7], on the contrary, presume a mereology of solids or spheres in order to define regions [3].

The apparent flexibility of taking concepts as primitive seems to be an inevitable characteristic of mental fictions [48] and logical reifications [34]. From a grounding perspective, however, the choice of primitives needs to be driven not by mathematical elegance in the first place, but by human perceptual competence. Similar to Greek mathematics, Suppes' proposal [42] uses constructive finite formalisms in order to deal with applied problems. In his formalism, geometric figures are explicitly constructed by a finite series of steps from a small point basis. The operations he proposes are *doubling* and *bisecting* of line segments, which allow, for example, to construct parallelograms (Fig. 1).

Note that this grounding approach differs from *finite geometry* in mathematics [21]. The interest is not in finite models for axiomatizations (such as affine plane figures of finite order), but one describes how geometric figures and properties (of a Euclidean flavour) can be constructed in finite sequences. This is not feasible in standard formalizations of Euclidean Geometry, since these require infinite models due to some of their axioms.

Infinity axioms[4] take the form of *universal-existential* sentences, i.e. , $\forall x_1, \ldots . \exists y_1 \ldots . \Phi(x_1, ..., y_1, ...)$. They allow to express recursions of existence claims, and thus to populate the domain of interpretation infinitely. Such axioms abound

[3] The theoretical basis of this idea is developed at some length in [36].

[4] By this notion, we vaguely refer to the axiomatic causes of infinity in a theory. These may resemble the *axiom of infinity* of ZF-set theory, which enforces a set containing successors for all its elements. We are yet unsure how to make our notion more precise, since universal-existential form is only a necessary criterion. However, we give examples for infinity axioms in the following.

in geometry and enforce their models to be infinite. If we take Tarski's elementary axiomatization of Euclidean Geometry [44], then we find 4 of the 13 axioms to be of such a form (or translatable into such a form, see [45]). For example, the *axiom schema of continuity (Axiom 13)* requires a boundary point for every two predicates that divide a ray into two halves. This is essentially the idea of a *Dedekind cut*, and thus requires the *continuum* of cardinality $2^{\aleph_0}$.

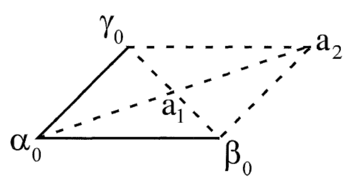

Fig. 1. Suppes' constructive geometry can be used to construct parallelograms. Bisecting line γ_0, β_0 yields new point a_1 and doubling line α_0, a_1 yields a_2. Compare [42].

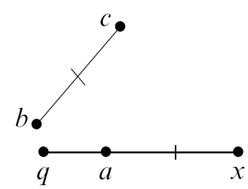

Fig. 2. Tarski's Axiom of Segment Construction. Compare [45].

But even if we dispense with Axiom 13, it is provable that models still need to be isomorphic to vector spaces over ordered fields [44], and these are infinite on the cardinality level $\aleph_0$. Reasons for this are the three remaining infinity axioms of *Pasch*, *Euclid* and the *Axiom of Segment Construction*. The latter axiom, for example, requires, for any existing pair of points b, c and for any given line (denoted by another pair of points, q, a), the existence of a pair of points a, x on that line which is congruent to b, c (compare Fig. 2). This requires infinity by itself: There is now a new pair of points on a line, for example q, x in Fig. 2. If we apply the axiom again to this pair and the line pair q, a, it requires a new pair a, x^* congruent to q, x, and so forth. Something equivalent is also enforced by the axiom of Pasch.

In a constructive geometry, infinity axioms like the axiom of segment construction need to be replaced by *explicit constructions*. These can be expressed in first order logic (FOL) by a finite list of existential quantifications that state the existence of any constructed point[5]. Another possibility is to describe the underlying operations explicitly, not only their results, in the spirit of Piaget's logic [27]. This can be done in terms of a constructive calculus [19], such as those used in intuitionistic logic or algebra. Both approaches are desirable and can be combined: In Sect. 4, we will use a constructive calculus in order to motivate a certain FOL axiomatization.

[5] This approach was taken in [36], and is based on Quine's proposal to express existence by the use of the existential quantifier [34].

3 Human Attention and Perceptual Operations

What kinds of human cognitive operations can serve as means of geometric construction? In [38] and [36], we have argued that the human attentional apparatus, through which human attention is anchored in pre-conceptual Gestalt mechanisms, can serve as the operational basis for semantic grounding[6]. The idea is that humans acquire certain pre-conceptual mechanisms in order to precompute Gestalts [15] in their perceived near-body environment. Gestalts serve as anchors of attention, i.e. they allow *referencing*, and enable one to predicate the presence of surfaces and other things (*perceptual predication*) without drawing on conceptual reasoning. The mechanisms may involve conscious parts and may be learned, e.g. in the sense of learning to play tennis: while the performance must be guided and learned consciously, the complex sensory-motor details are internalised.

The arguments for this view were recently advanced by Pylyshyn [31] based on empirical findings in object based attention [39]. He argued that without a pre-conceptual reference mechanism, human cognition would end up in a regress cycle of meaningless concepts. That this human attentional apparatus is at the same time the basis for inter-subjectivity of language, was recently argued by Tomasello [46] and is a central idea behind Quine's observation sentences [33]. According to Langacker [17] and Talmy, guided attention and Gestalt presence account for the meaning of language as such. We refer to the cited literature and to [38] and [36] for a deeper discussion.

3.1 Focusing Human Attention

We agree with von Glaserfeld[7], that there has to be some "pulsing" mechanism that produces *discrete* mental entities on the very lowest level of conscious perception[8].

We state the identity of a moment in which a human being focuses attention on a certain signal from the near-body environment. This signal may be a precomputed Gestalt, i.e., a structure pre-conceptually synthesized from visual, tactile, proprioceptive and other inputs, without the observer being necessarily aware of it. As any phenomenon, a Gestalt can enter consciousness only via *attentional moments*. The domain of foci of attention is considered as a root for other domains of consciousness, as it is the only one that can be directly coordinated across observers by the mechanism of joint attention [46]. It is also considered to be finite and therefore discrete, because human memory is bounded.

A *focus of attention* may be distinguished from an other because they come at different discrete time pulses. Perceiving time in its simplest form therefore

[6] A similar suggestion was made by Marchetti [22] and called 'attentional semantics'.

[7] Ernst von Glasersfeld developed a 'pulse' model for the mental construction of unities, pluralities and number; see, for example, Chapter 9 in [10] or [9].

[8] Although the question of whether conscious perception is discrete or not is still open, there is much psychophysical evidence for its discreteness [49].

means to perceive the *temporal order*, denoted by $\leq_T$, of foci of attention. The pulsing attention does not have to be focused on another signal. If it is, it produces a flow of conscious experience, which we call *perceptual predication.* Predication simply means that the human observer detects and stores the presence of Gestalts at one or several *foci of attention.* Mental operations are then available to construct higher level entities from this material flow of consciousness.

3.2 Identification of Point-like Features

It is essential to understand that the perception of *surfaces* plays a central role for many other kinds of perceptual operations that can be performed. In this spirit, Gibson [8] granted surfaces a central position in his ontology of the environment. We argue here for surface-based perceptual predication.

Observers identify prominent parts of their environment, such as relative parts of bodies, openings, or the free space in front of them, with respect to some already identified reference surfaces. These were called *features* in the DOLCE ontology [24]. They have their own criterion of identity, but existentially depend on an identifiable object, which is their "host". Perceivable features of a cup, for example, are its handle but also its opening. The opening of a cup would not exist without it, but is not a part of the cup. A feature of a building is the opening of its entrance. Further examples for features are the corner of a table or the peak of some mountain. We propose to call these latter examples *point-like features*, because they give rise to concentric sphere Gestalts that correspond to the mathematical fiction of a point.

Features are an important class of perceivable entities on their own, even though they depend on host surfaces. Studies in Gestalt psychology provide evidence for some sort of visual "hidden structure" that may account for this phenomenon. Rudolf Arnheim [2] studied the visual perception of balance, shape and form. He noticed that the perception of balance of black dots drawn into a square (Fig. 3) depends on how they are placed relative to the hidden field of visual tension shown in Fig. 4, which emerges relative to the square. Note that this field is not part of the square drawing. It rather depicts how black dots in the square are "dragged" towards its centers by a field of visual force. Arnheim assumes that this Gestalt mechanism accounts for the apparent human ability to detect whether the black dot is slightly off-center, without consciously comparing directions and lengths.

There is also recent evidence for a neurological mechanism underlying the intuitive sense of a location [4]. Burgess and others studied neurons in mammals, e.g. rats, that identify relative allocentric places (called *place cells*). These cells fire in response to other cells (called *boundary vector cells*), that detect surfaces at a certain allocentric direction and distance (see Fig. 5). Allocentric means that the firing of all these cells is independent of an egocentric reference frame, but depends on external landmark objects and surfaces [4]. There are even place cells configured in a grid-like manner [4]. Therefore, point-like features of this kind may be called "proto-locations". They can be considered preliminary stages of spatial reference frames. The space of kinesthetic coordination of our body

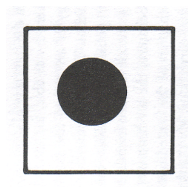

Fig. 3. A dot placed off-center into a square [2].

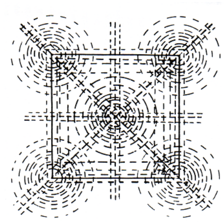

Fig. 4. The hidden field of visual force exterted on a dot placed into a square [2].

relies on lots of similar allocentric and egocentric mechanisms ([35], Chapter 3).

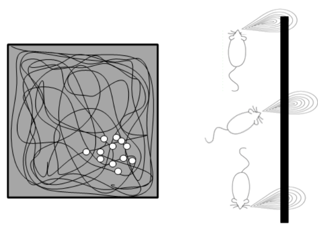

Fig. 5. Place cell firing (white dots) of a rat tracked in a box (left). Principle of boundary vector cells (right). Adapted from Burgess [4], see text for details.

We denote the predications of arbitrary point-like features by $PF(x, y)$[9]. The relation expresses that in the two moments of attention x and y, an observer focused on the same point-like feature.

3.3 Identification of Directions and Lengths

Since the operations of doubling or bisecting of lines [42] seem too restricted to capture constructive observed geometry in the near-body space, we will not directly use Suppes' proposal, even though we stick to his approach. Instead, we will develop a constructive modification of Tarski's axiomatization [44], because it is possible to interpret his primitives [45] in an intuitive way.

We proposed in [37] that humans experience the geometrical and topological structures of their environment by performing and comparing *attentional steps*. An attentional step is the *actual movement of attention* from focus x to y.

[9] Alternatively, one may want to differentiate different kinds of point-like features. One also may add another parameter for pointing at the reference surface of a feature, see [36].

Humans perceive *length* and *direction* of steps, because they are able to *compare steps of equal length and of equal direction*. And thereby, we assume, they are able to observe and measure lengths of arbitrary things in their environment.

We have suggested [36,37] that there are (at least) two Gestalt mechanisms available for geometric predication. One is a mechanism for comparing distances between pairs of foci. It can be conceived as the result of constructing a straight stick or some imagined straight Gestalt and being able to match its ends with two pairs of foci. For example, physically, we may align a stick with some object and move it around to match with some arbitrary foci of attention. We do exactly this when we use a non-collapsible compass. Note that the operation of comparing steps may be different from the one for performing steps[10]. The observation predicate $xy =_L uz$ (compare Fig. 6) asserts that foci x and y and foci u and z could be matched in this way[11].

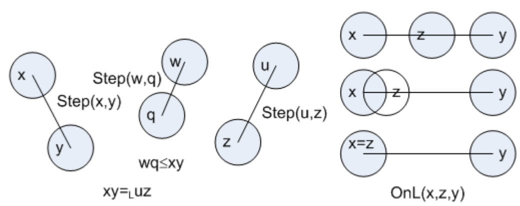

Fig. 6. Equal length and linear order for steps.

Another Gestalt mechanism allows for perceiving whether three foci of attention are ordered along a line[12]. $OnL(x, z, y)$ means that a focus z is *on a line* between x and y or co-located with any of them (compare Fig. 6). Note that OnL implies collinearity and betweenness[13]. It may be the result of comparing a focus of attention with two others by detecting whether or not it lies on an imagined line through them.

3.4 Identification of Locations

In distinction to common axiomatizations of point geometry, such as [44], the behavior of these observation predicates needs to be described not on the level of their domain and range, i.e., on the level of foci of attention, but with respect to constructed locations. The "points" of an experiential geometry are locations,

[10] We assume here that some operation for performing steps generates foci of attention, while some operation of comparing them generates geometric relations between them.

[11] This predicate was called 'congruence' by Taski in [44].

[12] Whether these foci were generated in a sequence of steps or not, is considered irrelevant here.

[13] This predicate was called 'betweenness' by Taski in [44].

not foci of attention. They exist only relative to a *frame of reference*[14] and certain comparison operations, as the ones introduced above. This distinction is ontologically essential, since a given attentional focus can be used to identify different locations with respect to different frames of reference. For example, if you sit in a train and focus two times on the apex of a table in front of you, then, at both moments, you are focusing *on the same point* with respect to the table, but on *two different points* with respect to a frame of reference located outside the train and being at rest relative to the landscape.

A reference frame is not only necessary to fix the measurement units and directions of an observed geometry. Together with basic comparison operations discussed above, it actually *establishes a geometry* with all its points and all its laws in the first place[15]. As A.S. Eddington argued, we must recognize "that all our knowledge of space rests on the behaviour of material measuring scales", and not on some pre-experiential absolute space [6].

To illustrate this argument, suppose you are sitting again in this train with some measuring tape at your disposal. Focusing on the table in front of you, you can jump with your attention from one of its ends x to another y and back to the first one x'. You will thereby notice that the length of the table has remained equal, i.e., $xy =_L yx'$, and therefore length comparison with reference to this frame and the tape is seemingly symmetric and suited for Euclidean geometry. But what if you choose a reference frame consisting of the table edges and a tree rushing past the window? If you jump with your attention from this tree to the table edge and back, symmetry of length is not preserved. So the choice of the frame of reference influences formal properties of your geometry. Similarly, if you choose to make length comparisons with a rubber band rather than a tape, symmetry properties may be preserved in the second case, but not in the first one (compare also [6]). So it is the choice of reference frame and comparison operations together that constitute an experiential geometry[16].

Euclidean-like geometry in perceived space can only be established based on choosing a stable reference frame of 4 *reidentifiable points* for 3 dimensions. Note that these four points need not only be reidentifiable by a single observer. If the geometry needs to be shared among people, the points also need to be indicated to others. From all we said above, this means they need to be chosen on the basis of shared Gestalts *external* to the geometrical system. We propose therefore that reference points may be based on point-like features, such as one identifiable corner of a perceived table.

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[14] We use the term frame here not in the sense of a formal reference system [16], but in the sense of perceivable point-like features one can refer to.

[15] This idea of relative space was proposed already by Leibniz [18] and Poincaré [28].

[16] But nevertheless, these choices do not yet determine it, as Poincaré argued [28]. Geometry is likewise affected by Quine's empirical indeterminacy [32], in the sense that, given a reference frame and comparison operators, there is more than one way of building a geometry.

4 An Operational Model of Constructive Relative Geometry

Our goal is to show how observations expressed by the predicates OnL and $=_L$, as well as relative locations, may be constructed by a human observer. Our main argument of Sect. 2 was that in a constructive finite geometry, infinity axioms need to be substituted by explict constructions. But how to describe an explicit construction in a formal way? We argued that one possibility is to describe the underlying operations explicitly in terms of an *operational calculus*. This means to describe experiential geometry by way of the operations that may generate it. In the next subsection, we will discuss the notion of an operational calculus known from intuitionism, and suggest how it may be reused to do finite constructions as well as geometric inference by concatenating constructive and inference calculi. We also show that locations or "points" of a geometry need not be presumed but can be constructed relative to a reference frame.

4.1 Operational Calculi, Inference and Explicit Construction

We suggest to use a form of Paul Lorenzen's operational calculus [19] to describe an operational model of the perceptual operations that were discussed in the last section. Note that a calculus, which is formal, should not be confused with the actual human operations it describes[17]. Furthermore, it only reflects our own preliminary ideas, which may need revision in the future[18].

An operational calculus (not to be confused with infinitesimal calculus) is a basic mathematical tool of formal construction based on rules. Its most prominent application areas are formal inference in logic, where the validity of a sentence is proved by generating it from other ones using certain rules of deduction, and the formation of well-formed sentences from syntactic atoms. In intuitionism, constructive calculi are used in a more fundamental way, namely as a tool of constructive justification for logic as such, based on the idea of inductive proofs. Starting with Brouwer, Heyting and Kolmogorov, intuitionists have clarified the meaning of logical constants as well as inference rules based on operating with certain calculi[19]. Lorenzen's general conception of a calculus does not only apply to logic or inference, but also to mathematical object construction. For example, the mathematical idea of infinity can be reduced to potential infinity if we conceive it in terms of a calculus [20]. The flexibility of a calculus allows us to do both, constructing objects as well as to infer facts about them.

A calculus is, according to Lorenzen [19], a description of a procedure to generate symbols ("Figuren") from given symbols. The given symbols are written

[17] In particular, we do not claim here that cognitive human operations *are* formal symbol manipulations, as claimed in [26].

[18] In particular, an important further development will be to attempt a calculus without predicates at all (other than equality), i.e., an algebra of perceptual operations.

[19] Lorenzen's "protologic" [19] can be seen as an early attempt to give logical constants and deduction rules a proof-theoretic semantics, similar to Prawitz [29] and Dummett [5] (see [40]).

down at the *beginning* (A). New symbols are generated using a set of *rules*[20] (R) that can be iteratively applied to symbols. The rules have free variables standing for symbols to be substituted for them. For example, the following is a calculus for the construction of natural numbers:

K_{nat} :

1	(A)	(primitive atom: 1)
$x \longmapsto x1$	(R)	(object variable: x)

Note that the arrow above is not a logical implication but denotes a rule, i.e. the permission to write down an instance of the symbols at the end of the arrow for every substitution of variables with objects. We call these variables *object variables* and denote them by lower case letters $x, y, z....$ They stand for objects constructible in the calculus, in this case for numbers. A rule can have more than one input or output figure separated by a comma, $I, I, ... \longmapsto O, O, ...$ At the beginning of this calculus, there only exists the *primitive atom* 1. If we iteratively apply rule R starting from the atom, we can generate a *series of strings of primitive atoms*, e.g, $R(1) = 11$, $R(R(1)) = 111$, ... These strings of primitive atoms are called *objects*, whereas strings of primitive atoms with object variables, e.g. $x1$ in the rule above, are called *object formulas*. We generate new objects by inserting objects into object formulas. We call a set of symbols generated in this way a *derivation*. Note that derived objects can always be ordered according to their production sequence.

We not only need to generate objects, but also realize relations among them. *Predications* are strings of objects and relation symbols generated according to further rules. *Formulas* are strings that additionally contain object variables and are used in these rules. We generate predications by substituting objects in formulas.

K_{nat+} :

$1 + 1 = 11$	(A)	(primitive atoms: 1, +, =)
$x + 1 = y \longmapsto x1 + 1 = y1$	(R_1)	(object variables: x, y, z)
$x + y = z \longmapsto x + y1 = z1$	(R_2)	

For example, arithmetics can be constructed by rules R_1 and R_2, which allow to derive the predication $11 + 11 = 1111$ by substituting objects in the formula $x + y = z$. Note that in contrast to Tarskian formal semantics, operative figures do not have an interpretation into a domain. The distinction between predicates and objects is therefore not based on such an interpretation.

Let us return now to our initial question: What is an explicit construction? Our calculus K_{nat+} obviously contains closure rules. Our constructed arithmetic set would be infinite if every constructible object actually was constructed, but this is impossible as a matter of fact. We suggest therefore that an explicit construction is not a calculus, but involves a particular *application of a calculus*

[20] These correspond to axioms and theorems in axiomatic theories.

in a finite number of steps, i.e. a particular derivation. This requires that the *existence* of constructed entities needs to be conceived in terms of *derivation*, not derivability.

In an intuitionist sense, however, existence $\exists$, as well as other logical constants, are based on *derivability* in a calculus. Here, $\exists x$ just means that object x with certain properties *can* be derived [19]. Similarly, inference of rules, called *admissibility* by Lorenzen [40], is based on derivability: A rule is called admissible if it does not increase the set of derivable figures. For example, a rule is admissible if the head of the rule can be derived from the condition by concatenating already admissible rules (deduction)[21]. Analogously, *negation* is understood in terms of underivability: $\neg A$ means that predication A is underivable in the calculus. This can be expressed by a particular admissible rule: $\neg A$ is defined as the rule $A \longmapsto \perp$, where $\perp$ is an underivable symbol in the respective calculus [19][22]. Using this definition we can also infer negations, for example by *reductio ad absurdum (R.A.A.)*: Suppose we have already derived $\neg A$ and $B \longmapsto A$, where B is the hypothesis to be refuted. It is then easy to infer that $\neg B$. For, since by condition $B \longmapsto A$ and by definition $A \longmapsto \perp$, we have $B \longmapsto \perp$ by deduction, which just means $\neg B$.

We intend to use an operative calculus not only to do logical inference, but also to generate initial finite models. Since the former is based on derivability while the latter on actual derivation, we propose to distinguish calculi according to their *purpose*, i.e. between *inferential* and *constructive* calculi. A constructive calculus is just an auxiliary mechanism to construct a finite domain. Non-existence means that an object actually has not been constructed in a particular derivation, which encompasses but is broader than underivability. This interpretation is useful since it can reflect a *particular observation process*: Observation is a product of an observer taking perceptual decisions and directing his attention at phenomena present in his field of view. Thereby, he does not everything he may be able to do, and we are only interested in his observed facts.

In an inferential calculus, instead, just as in intuitionistic logic, objects exist and predications are true if and only if they *are derivable* in it. True facts are either given (i.e., observed) or derivable from the given, and all others are considered false. If the constant $\perp$ has been introduced into the calculus, negations can be inferred by showing underivability from observed facts[23]. Regarding human perception, such a calculus has its analogue in *Gestalt completion mechanisms* [15], which account for large parts of unconscious automatic human reasoning.

In order to generate a model of finite relativist geometry, we propose therefore to concatenate operational calculi in the following way: In Sect. 4.2, we generate the domain of foci of attention in a constructive calculus, in which object deriva-

[21] There are still further inference principles, e.g. inversion, see [19].

[22] If this rule is admissible and $\perp$ is underivable, then A must be underivable, too.

[23] Another possibility is that the observer may predicate negative statements directly, i.e. in terms of *observed absence of a phenomenon in the field of view*. In order to account for the existence of occluded phenomena, observed absence thereby does not imply non-existence.

tion (not: derivability) corresponds to an observer's generation of a focus, and which allows "positive" initial predications as observed facts. In such a calculus, contradiction can never occur since every generated statement is a positive assertion of some observed phenomenon. In order to infer expected Euclidean properties from these observed facts, we then apply an inferential calculus in the standard intuitionist sense to the finite number of facts generated in the former calculus. This *relational closure calculus* contains closure rules (Sect. 4.3) which largely correspond to geometric implications. The set of predications derivable in this way is finite, because the set of relations on a finite domain is bound to be finite. Note that the input to the inferential calculus are those and only those positive facts generated in one particular derivation of the constructive calculus. Furthermore, closure rules may have negations in their condition but not in their heads. Thus the resulting model is assured to be finite and does not contain contradictions.

We are aware that this procedure does not yet ensure that every model derivable in this way is one of the intended finite relativist geometry. Our paper goal is rather to construct one such model, and use it to demonstrate the consistency of constructive inference rules.

4.2 Initial Attentional Construction

In the following, we will use letters $a, b, c, d, ..., t$ to denote objects and x, y, z, v, w, u to denote object variables. For convenience, we will use the following abbreviation for object figures: Let o be an object atom with $a_0 \equiv o$. Let x_y stand for any object constructed by concatenation, then $x_{y+1} \equiv x_y o$. For example, $a_1 \equiv a_0 o \equiv oo$. Thus, two object figures with different increments are always unequal. Now consider the following calculus for generating attentional steps and their observed interrelations:

$$
\begin{array}{lll}
K_{attention} : \text{(primitive atoms: o, } OnL, =_L, PF) & & \\
& a_0 & (A) \\
& x_y \longmapsto x_{y+1} & (R_{step}) \\
& x_y, x_w, x_u, x_v \longmapsto x_y x_w =_L x_u x_v & (R_L) \\
& x_y, x_w, x_u \longmapsto OnL(x_y, x_u, x_w) & (R_{OnL}) \\
& x_y, x_w \longmapsto PF(x_y, x_w) & (R_{PF})
\end{array}
$$

The beginning of this calculus consists of a single focus of attention a_0. The first rule is a rule to generate a new attentional moment starting from any attentional moment, i.e., to take a step. It generates foci of attention by incrementing the subscripts of their symbols. All other rules do not generate foci of attention, but relations among them. The second rule generates a predication stating that x_u, x_v has the same distance as x_y, x_w. The third rule generates a predication stating that x_u lies on the line between x_y, x_w. The fourth rule associates two foci on the same point-like feature.

Taking an attentional step stands for the *act of referencing*, i.e. for moving the attentional focus in the near-body environment. It generates our domain of interpretation. The other operations represent *perceptual predications*, i.e. operations that compare and associate foci of attention with each other. In order to take an attentional step with specific properties, we have to construct it by *concatenating operations*, i.e. by first taking a step and then generating the required relations by comparison. This reflects the idea that predication and referencing are two different kinds of operations, while in practice they may happen almost synchronically.

In this calculus, we can generate the following sequence of foci of attention constructing an *equilateral triangle* (compare Fig. 7). We first make a step from the beginning to any other focus. Then we add another step such that the pairs consisting of new focus and the two previous ones are congruent to the first step.

	$Derivation_{Triangle}$:	
(1)	a_1	$\vert R_{step}(a_0)$
(2)	a_2	$\vert R_{step}(a_1)$
(3)	$a_0a_1 =_L a_1a_2$	$\vert R_L(a_0, a_1, a_1, a_2)$
(4)	$a_0a_1 =_L a_0a_2$	$\vert R_L(a_0, a_1, a_0, a_2)$

This is basically a description of what we do when we construct a triangle with a compass, where a_2 is in the intersection of two circles centered on a_0 and a_1 with radius a_0, a_1. Note that this procedure generates a nondegenerate triangle only if these foci do not coincide (and are not on a line). This means we first need to construct the notion of locational coincidence, which exists only relative to some frame of reference.

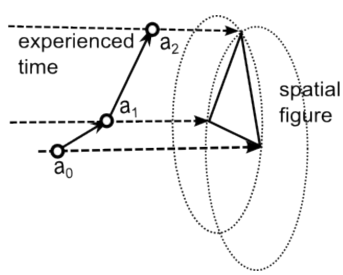

Fig. 7. Construction of a triangle in terms of attentional moments. Time is indicated by the horizontal 3rd dimension in this figure.

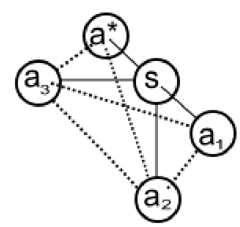

Fig. 8. A reference frame for a 3-dimensional cartesian coordinate system. The figure depicts its 3-dim spatial projection without time.

A *spatial reference frame* consists of 3 point-like features standing perpendicular to each other on a common origin, which is also a point like feature. For better readability, we write focus names as depicted in Fig. 8 and not with proper increments.

	$Derivation_{ReferenceFrame}$:	
(1)	s	$\|R_{step}(a^*)$
(2)	$PF(s,s)$	$\|R_{PF}(s,s)$
(3)	a_1	$\|R_{step}(s)$
(4)	$PF(a_1,a_1)$	$\|R_{PF}(a_1,a_1)$
(5)	$a^*s =_L sa_1$	$\|R_L(a^*,s,s,a_1)$
(6)	$OnL(a^*,s,a_1)$	$\|R_{OnL}(a^*,s,a_1)$
(7)	a_2	$\|R_{step}(a_1)$
(8)	$PF(a_2,a_2)$	$\|R_{PF}(a_2,a_2)$
(9)	$a^*s =_L sa_2$	$\|R_L(a^*,s,s,a_2)$
(10)	a_3	$\|R_{step}(a_2)$
(11)	$PF(a_3,a_3)$	$\|R_{PF}(a_3,a_3)$
(12)	$a^*s =_L sa_3$	$\|R_L(a^*,s,s,a_3)$
(13)	$a^*a_2 =_L a_2a_3$	$\|R_L(a^*,a_2,a_2,a_3)$
(14)	$a^*a_2 =_L a_1a_3$	$\|R_L(a^*,a_2,a_1,a_3)$
(15)	$a^*a_2 =_L a_1a_2$	$\|R_L(a^*,a_2,a_1,a_2)$

In this construction, s denotes a focus on the origin and a_i denotes one of three foci on *perpendicular unit vectors* in this reference system (compare Fig. 8). Focus a^* is on an auxiliary point needed to assert orthogonality. Orthogonality is assured by the condition that distances of foci a_i to each other are all congruent to a^*a_2, and by the fact that a^* is not on the same point-like feature as s[24]. This latter negative fact is inferred in the inference calculus of Sect. 4.3.

The construction of this reference frame assures that our model has got *at least 3 spatial dimensions*. It therefore directly corresponds to Tarski's *lower dimension axiom* for 3D (compare [45]). Note also that a primitive way of time perception ($\leq_T$) is given by the derivation order.

4.3 Relational Closure Calculus

The following inferential calculus is intended to close the domain of relations with respect to perceptual predications in such a way that it reflects our expectations about experienced geometry. The rules that have to be introduced largely correspond to geometric axioms in a FOL theory, such as the one of Tarski [41].

[24] Orthogonality can then be proved along the following lines (compare Fig. 8): a^*, s, a_1 lie on distinct point-like features of a line. Thus angle a^*, s, a_2 must be supplementary to angle a_1, s, a_2. Since segment a^*a_2 is congruent to a_1a_2, and the angle sides must also be congruent by construction, triangles a^*, s, a_2 and a_1, s, a_2 must be congruent, too. Thus the supplementary angles must be congruent. The intended result now follows from the fact that congruent supplementary angles are always right angles.

Yet, our calculus assures finite constructibility and accounts for relativity of locations.

The beginning of this calculus consists of all and only those objects and facts generated by the *reference frame derivation* of the initial attentional calculus in the last section[25]. The calculus exclusively contains rules to add new relation tuples based on existing ones (relational closure rules), so the object domain remains equal. We will use inference to show which Euclidean properties are entailed by this calculus. We will also show that *relative* geometry, which is built on foci of attention, not locations, behaves neutral with respect to foci on the same location, as expected.

Reference Frame Rules As argued in Sect. 3.4, a spatial reference frame consists of point-like features, not foci of attention, and can be used to define locations relative to it. For this purpose, it needs to retain the intrinsic geometric properties constructed in the last section through time. This can be expressed by requiring the same configuration for arbitrary foci that lie on the same point-like features (we use $\bigwedge$ for iterating over inputs and outputs). For a rule that can be used into both directions, we write $\rightleftharpoons$. We first abbreviate the fact that foci lie on the same reference frame:

Rule 1 $(D_{Ref}) : PF(s, x_s), \bigwedge_{i=1}^{3} PF(a_i, x_i) \rightleftharpoons RefFrame(x_s, x_1, x_2, x_3)$

Rule 2 $(R_{Ref}) : RefFrame(x_s, x_1, x_2, x_3) \longmapsto OnL(a^*, x_s, x_1), \bigwedge_{i=1}^{3} a^* s =_L x_s x_i,$
$\bigwedge_{1 \le i < j \le 3} x_i x_j =_L a_1 a_2$

Rule 3 $(R_{fix}) : RefFrame(x_s, x_1, x_2, x_3) \rightleftharpoons x_s s =_L ss, \bigwedge_{i=1}^{3} x_i a_i =_L a_i a_i$

This last rule says that foci on the same point-like feature of the frame have zero distance from each other. This assures that the reference frame is immovable with respect to the observer. It is later used to prove that the four point-like features correspond to four locations. Note that from rule 1 and the reflexive facts about PF in the reference frame construction (p. XVf.), it follows immediately that $RefFrame(s, a_1, a_2, a_3)$.

Rules about Congruence Consider the following closure rules for $=_L$:

Rule 4 $(Reflexivity_{=_L}) : x, y \longmapsto xy =_L xy, xx =_L yy$

Rule 5 $(Connectivity_{=_L}) : xy =_L zu, xy =_L vw \longmapsto zu =_L vw$

Rule 6 $(Identity_{=_L}) : xy =_L zz \longmapsto x =_{Ref} y$

These rules seem to comply with our intuition about length comparison: The distance of two foci is always equal to itself, and if a length equals two other lengths, then those are equal, two. The last rule says that the distance of a focus to itself is the same as the distance between two foci on the same location, denoted by $x =_{Ref} y$. The following rules can immediately be proven to follow:

[25] The object constants s, a^*, a_1, a_2, a_3 denote the generated foci of attention.

Derived rule 1 *(Symmetry$_{=_L}$):* $xy =_L zu \longmapsto zu =_L xy$

Proof. $xy =_L zu$ by condition and $xy =_L zu$ by rule 4. By rule 5, $zu =_L xy$.

Derived rule 2 *(Transitivity$_{=_L}$):* $xy =_L uv, uv =_L vz \longmapsto xy =_L vz$

Proof. Applying derived rule 1 to the first input above directly yields rule 5.

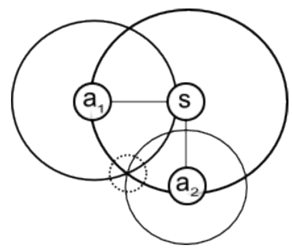

Fig. 9. The location of a focus relative to the frame is fixed by its distance to the point-like features of the frame.

In order to express that our model has got no more than three spatial dimensions, we additionally require that the relative location of two foci of attention x, y is fixed once their distances to the chosen four point-like features of the reference frame are equal (Fig. 9). This is done by the following rules, which define *location equivalence* relative to the frame introduced above:

Rule 7 (D_{locus}) : $\mathit{RefFrame}(x_s, x_1, x_2, x_3), xx_s =_L yx_s, \bigwedge_{i=1}^{3} xx_i =_L yx_i \longmapsto x =_{Ref} y$

Rule 8 (R_{3D}) : $x =_{Ref} x', y =_{Ref} y' \longmapsto xy =_L y'x'$

Location equivalence $x =_{Ref} x'$ simply means that x and x' are bound to have the same distances to every other location in focus on y and y'. To put it in another way: length comparison behaves neutrally with respect to foci on the same location. Note that the last rule has y' and x' reversed, which can be used to prove that the order of foci is irrelevant. This is because equidistant steps are reversible: it is always possible to return to the same locus by taking a step forward and then a step back with the same length.

Derived rule 3 *(Reflexivity$_{=_{Ref}}$):* $x \longmapsto x =_{Ref} x$

Proof. By rule 4, we have $xa =_L xa$, where $a \in \{s, a_1, a_2, a_3\}$. By rule 7, therefore $x =_{Ref} x$.

Derived rule 4 *(Symmetry$_{=_{Ref}}$):* $x =_{Ref} y \longmapsto y =_{Ref} x$

Proof. By derived rule 1 and the input, we have $ya =_L xa$ with $a \in \{s, a_1, a_2, a_3\}$. So by rule 7, $y =_{Ref} x$.

Derived rule 5 (*Reversibility*$_{=_L}$)*:* $x, y \longmapsto xy =_L yx$

Proof. By derived rule 3, $x =_{Ref} x$ and $y =_{Ref} y$. By rule 8, $xy =_L yx$.

The challenge is now to show that from these rules, a general *locus neutral* geometry on foci can be obtained. The following generalizes these results over location equivalent foci:

Derived rule 6 *(Locus neutrality 1):* $x =_{Ref} x', y =_{Ref} y' \longmapsto xy =_L x'y'$

Proof. By rule 8 and derived rule 3, we have $xy =_L yx$ and $y'x' =_L x'y'$. By rule 8 and the input, it follows also that $xy =_L y'x'$. By derived rule 2, then $xy =_L x'y'$.

Derived rule 7 $x =_{Ref} y \longmapsto xx =_L xy$

Proof. By derived rule 3, we have $x =_{Ref} x$. By input also $x =_{Ref} y$, and so, by derived rule 6, $xx =_L xy$.

Derived rule 8 *(Locus neutrality 2):* $x =_{Ref} x', y =_{Ref} y' \longmapsto xx' =_L yy'$

Proof. By derived rule 7 and the input, we have $xx =_L xx'$ and $yy =_L yy'$. By rule 4, we also have $xx =_L yy$, and by derived rule 2, $xx =_L yy'$.

In the following, *for convenience of reading, if we use object variables* u, u' *with prime, we implicitly consider* $u =_{Ref} u'$ *among the conditions of the respective rule*:

Derived rule 9 *(General connectivity):* $xy =_L zu, x'y' =_L vw \longmapsto z'u' =_L v'w'$

Proof. $xy =_L zu$ by condition. Since also $zu =_L z'u'$ by locus neutrality 1, we get $z'u' =_L xy$ by connectivity and symmetry. With $x'y' =_L vw$ by condition and $vw =_L v'w'$ by locus neutrality 1, we get $x'y' =_L v'w'$ by transitivity. Using $xy =_L x'y'$, we get the required result by transitivity.

Taking a step of zero length, i.e., to step on the spot, leads to the same locus:

Derived rule 10 *(Locus identity):* $xy =_L zz' \rightleftharpoons x =_{Ref} y$

Proof. From right to left: $x =_{Ref} y$ and $z =_{Ref} z'$ by conditions. By locus neutrality 2, the result immediately follows. From left to right: $z =_{Ref} z'$ by condition, so $zz =_L zz'$ by derived rule 7. By condition, also $xy =_L zz'$, and so $xy =_L zz$ by connectivity. By rule 6, we get $x =_{Ref} y$.

If we substitute location equivalence $=_{Ref}$ with the identity sign[26] $=$, then derived rule 10 corresponds to Tarski's identity axiom of congruence [41,45], rule R_{3D} to Tarki's first congruence axiom [41,45], and derived rule 9 to congruence Axiom 2 in [41,45]. We have already mentioned that the lower dimension axiom [41,45] is captured by the construction of the reference frame itself. Note that locus identity is a biconditional instead of a simple implication as in [41]. This directly assures that steps of zero length are always congruent to each other, without any need to draw on the Axiom of Segment Construction.

[26] But note that $=$ in our theory does not mean the same as $=_{Ref}$, because foci are not locations.

Rules about Point-like Features Now consider the following rules which basically generate the *symmetric transitive closure* of PF:

Rule 9 *(Reflexivity$_{PF}$):* $PF(x, y) \longmapsto PF(x, x), PF(y, y)$

Rule 10 *(TransSym$_{PF}$):* $PF(x, y), PF(y, z) \longmapsto PF(x, z), PF(z, x)$

We can prove now that all foci on the same point-like feature of the chosen reference frame (which are an equivalence class) are also on the same location. This means the point-like features of the frame must range among the locations:

Derived rule 11 $PF(x, s), PF(y, s) \longmapsto x =_{Ref} y$

Proof. By rule 1, we have $RefFrame(x, a_1, a_2, a_3)$ and $RefFrame(y, a_1, a_2, a_3)$. By rule 2, we have $a^*s =_L xa_i$ and $a^*s =_L ya_i$ for all $i \in \{1, 2, 3\}$. By transitivity, $ya_i =_L xa_i$, and by rule 3 and rule 6, $xs =_L ys$. This satisfies the input of rule 7, and so $x =_{Ref} y$.

Rules about Collinearity and Order The following rules characterize the collinearity relation OnL. These are quite numerous compared to [41], because we have to compensate for the loss of the Axiom of Pasch and Segment Construction, compare [41]. First, OnL also has some identity rule: If we focus our attention on a spot lying between two foci located at the same locus, then this spot must be at the very same locus. The reflexivity axiom assures that OnL applies to the degenerate case where two points coincide, and symmetry captures the comprehensible fact that points on a line can be ordered in two ways. The other rules assure that OnL-triples with two points in common are ordered on a line, as one would expect.

Rule 11 *(Identity$_{OnL}$):* $OnL(x, y, x'), x =_{Ref} x' \longmapsto x =_{Ref} y$

Rule 12 *(Reflexivity$_{OnL}$):* $y =_{Ref} y', x \longmapsto OnL(x, y, y')$

Rule 13 *(Symmetry$_{OnL}$):* $OnL(x, y, z) \longmapsto OnL(z, y, x)$

Rule 14 *(InnerTrans$_{OnL}$):* $OnL(x, y, z), OnL(y, u, z) \longmapsto OnL(x, y, u), OnL(x, u, z)$

Similarly, it is now possible to generalize these results over location equivalent foci[27]:

Derived rule 12 *(Locus neutrality 3):* $OnL(x, y, z) \longmapsto OnL(x', y', z')$

Proof. $OnL(x, y, z)$ by condition and $OnL(y, y', z)$ by reflexivity and symmetry. By rule 14, we obtain $OnL(x, y, z')$, and reversing the result by symmetry yields $OnL(z', y, x)$. Together with $OnL(y, y', x)$ by reflexivity, we get $OnL(z', y', x)$ by rule 14. And again, since $OnL(y', x', x)$ by reflexivity and symmetry, we obtain $OnL(z', y', x')$ by rule 14, and so $OnL(x', y', z')$ by symmetry.

[27] Remember that the condition $x =_{Ref} x'$ is abbreviated using primed variables x, x'.

Derived rule 13 *(General Inner Transitivity):*
$OnL(x, y, z), OnL(y', u, z') \longmapsto OnL(x', y', u'), OnL(x', u', z')$

Proof. Follows immediately from applying rule 14 and derived rule 12.

Derived rule 14 $OnL(x, y, z), OnL(x', z', v) \longmapsto OnL(y', z', v'), OnL(x', y', v')$

Proof. If we convert the condition by symmetry, we get the condition for derived rule 13, and thus $OnL(v', z', y')$, whose symmetrical conversion yields the first required result. Similarly, for the second result.

If we substitute $=_{Ref}$ with $=$, then rules 11, 12 and 13, as well as derived rules 13 and 14 correspond to essential axioms or theorems in [41]. The rules corresponding theorems cannot be derived as in [41] because of the loss of infinity axioms.

Subtractivity of Lengths Now we need to add rules for governing the interrelation of the two geometrical observation predicates. These are essential in order to describe something similar to a Euclidean space. It turns out that we can use variants of Tarski's five segment axiom in order to obtain a finite version of *absolute geometry.*

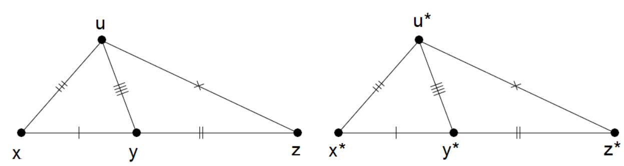

Fig. 10. The (Inner) Five Segment Axiom. The length of segment yu is fixed once x, y, z, u exhibit a five segment configuration. Source: [45].

The so called *(inner) five segment axiom* allows us to express length summations as well as to characterize angles. As shown in Fig. 10, the rule states that in a certain configuration of four segments, the length of a fifth segment needs to be fixed, i.e.

Rule 15 (*Inner5Seg*)*:* $IFS\left(\begin{smallmatrix} x & y & z & u \\ x^* & y^* & z^* & u^* \end{smallmatrix}\right) \longmapsto yu =_L y^*u^*$

, where $IFS\left(\begin{smallmatrix} x & y & z & u \\ x^* & y^* & z^* & u^* \end{smallmatrix}\right)$ abbreviates $OnL(x, y, z), OnL(x^*, y^*, z^*), xz =_L x^*z^*$, $yz =_L y^*z^*, xu =_L x^*u^*, zu =_L z^*u^*$.

For example, using this rule, we can prove a subtractivity property of lengths: Subtracting congruent segments from congruent segments derives congruent segments:

Derived rule 15 *(Subtractivity):*
$OnL(x, y, z), OnL(x^*, y^*, z^*), xz =_L x^*z^*, yz =_L y^*z^* \longmapsto xy =_L x^*y^*$

Proof. Apply rule *Inner5Seg* to the condition taking x for u.

Inferring and Using Negations The facts inferred so far do not incorporate any negative statements. We show lastly how to derive negative statements from observed positive facts in an intuitionist manner, and how to use these as inputs to further rules that have negations in their conditions.

We can infer that our reference frame needs to be non-degenerate, by proving that two foci of the reference frame do not coincide:

$$\neg a^* =_{Ref} s \tag{16}$$

Proof. We first prove the underivability of $PF(a^*, s)$ in our inference calculus. By rules 9 and 10, point-like features are equivalence classes of foci, so $PF(a^*, s)$ would be the case if and only if a^* was in the same equivalence class that contains s. The latter is just $\{s\}$ by the reference frame derivation step (2) on page XV. Since there are no other inference rules to derive $PF(a^*, s)$, it is admissible that $PF(a^*, s) \longmapsto \bot$. Now we prove the rest by R.A.A. By derived rule 7, we have $s =_{Ref} a^* \longmapsto a^*s =_L ss$. Applying reflexivity rule 4 on all a_i, we can use rule 3 (inverse direction) to derive $s =_{Ref} a^* \longmapsto \mathit{RefFrame}(a^*, a_1, a_2, a_3)$, and rule 1 (inverse direction) to obtain $s =_{Ref} a^* \longmapsto PF(a^*, s)$. With $PF(a^*, s) \longmapsto \bot$, we obtain $s =_{Ref} a^* \longmapsto \bot$, and this just means $\neg a^* =_{Ref} s$ by definition.

It can now be similarly proved by contradiction that the point-like features of the frame must not be coincident, too, e.g. for a pair x_s, x_1:

Derived rule 16 *(Non-degeneracy):*
$\mathit{RefFrame}(x_s, x_1, x_2, x_3) \longmapsto \neg x_s =_{Ref} x_1$

Proof. From the condition and from rule R_{Ref}, we know that $a^*s =_L x_s x_1$. Now suppose $x_s =_{Ref} x_1$ was derivable by inference. Then, by derived rule 10 (from left to right), this would mean $a^* =_{Ref} s$. But this would contradict the already derived statement $\neg a^* =_{Ref} s$ above.

Once negation is introduced, we may add rules that have negations in their conditions (negative closure rules). For example, there is one rule misssing in order to derive that OnL triples with two loci in common are ordered on the same line (compare [41]):

Rule 16 *$\mathit{OuterTrans}_{OnL}$:*
$OnL(x, y, z), OnL(y', z, u'), \neg y =_{Ref} z \longmapsto OnL(x', y', u'), OnL(x', z', u')$

We can then also add a variant of the five segment axiom with negative condition in order to assure additivity of segments [41].

5 Conclusion

We have argued for certain perceptual operations that can be used to ground geometry experientially. The human attentional apparatus allows for referencing and predication of geometrically relevant Gestalt phenomena in the vista environment. In particular, it allows for detecting whether one focus of attention precedes another one (primitive perception of time), whether attention focuses on the same point-like feature (PF), whether a given pair of foci is congruent to another pair ($=_L$), and whether a focus points between two others (OnL). We argued further that, in order to construct a geometry in the usual relativist sense based on these human competences, we need to identify a frame of reference made of point-like features, with respect to which relative locations, i.e. points, can be identified by arbitrary observers.

We have used a constructive calculus to generate such a frame of reference. A further inferential calculus with closure rules allowed us to introduce location equivalence $=_{Ref}$ with respect to this frame, and to derive much of the intuitively expected behavior of the geometric notions. In particular, it allows to derive and use negations in an intuitionist sense. The former calculus was only used to generate a particular initial construction, which corresponds to conscious attentional selection and perceptual predication. The latter one accounts for geometric inference, understood here as a kind of automatic Gestalt completion. In this way, we constructed a reference frame that corresponds to the 3D lower dimension axiom and its geometric properties by a set of inference rules (rules 1-17) that correspond to axioms and definitions of a FOL geometry. Such a theory is a finite relativist variant of *absolute geometry*, which does not have the parallel postulate, but allows us to define angles, lengths and projections with their usual properties (compare [41], and also [36]), and furthermore to define locations relative to the reference frame. A simple model of this theory can be generated by exhaustively applying the inference calculus to the initial finite construction.

Regarding the two geometric calculi, it would be desirable to have a consistent set of rules that ensure every derivation to be a model of finite relativist geometry. This is not the case yet. It is, for example, possible to construct a $> 3D$ initial space which would not comply with R_{3D}. Regarding completeness, the calculus lacks a rule that corresponds to the disjunctive outer connectivity Axiom 18 in [45], since intuitionist disjunctions require one of the disjuncts to be derivable [19]. In general, it is open which of the remaining Euclidean axioms in [45] (e.g. the parallel postulate) can be given such a constructive interpretation.

Acknowledgements

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